

WELL-POSEDNESS OF AN INTEGRO-DIFFERENTIAL EQUATION WITH POSITIVE TYPE KERNELS MODELING FRACTIONAL ORDER VISCOELASTICITY

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ABSTRACT. A hyperbolic type integro-differential equation with two weakly singular kernels is considered together with mixed homogeneous Dirichlet and non-homogeneous Neumann boundary conditions. Existence and uniqueness of the solution is proved by means of Galerkin's method. Regularity estimates are proved and the limitations of the regularity are discussed. The approach presented here is also used to prove regularity of any order for models with smooth kernels, that arise in the theory of linear viscoelasticity, under the appropriate assumptions on data.

1. INTRODUCTION

We study the model problem (2.6), which is a hyperbolic type integro-differential equation with two weakly singular kernels of Mittag-Leffler type. This problem arises as a model for fractional order viscoelasticity. The fractional order viscoelastic model, that is, the linear viscoelastic model with fractional order operators in the constitutive equations, is capable of describing the behavior of many viscoelastic materials by using only a few parameters.

A perfectly elastic material does not exist since in reality: inelasticity is always present. This inelasticity leads to energy dissipation or damping. Therefore, for a wide class of materials it is not sufficient to use an elastic constitutive model to capture the mechanical behaviour. In order to replace extensive experimental tests by numerical simulations there is a need for an accurate material model. Therefore viscoelastic constitutive models have frequently been used to simulate the time dependent behaviour of polymeric materials. The classical linear viscoelastic models that use integer order time derivatives in the constitutive laws, require an excessive number of parameters to accurately predict observed material behaviour.

Bagley and Torvik [3] used fractional derivatives to construct stress-strain relationships for viscoelastic materials. The advantage of this approach is that very few empirical parameters are required. When this fractional derivative model of viscoelasticity is incorporated directly into the structural equations a time differential equation of non-integer order higher than two is obtained. One consequence of this is that initial conditions of fractional order higher than one are required. The problems with initial conditions of fractional order have been discussed by Enelund and Olsson [9], see also references therein. To avoid the difficulties with fractional

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order initial conditions some alternative formulations of the fractional derivative viscoelastic model are used in structural modeling. The formulation that we use, is based on a convolution integral formulation with a singular kernel of Mittag-Leffler type, see [1], [8], and [9].

There is an extensive literature regarding well-posedness and numerical treatment for integro-differential equations, see, e.g., [1], [4], [11], [12], [13], [14], [15], [16], [17], and [18]. Existence, uniqueness and regularity of a parabolic type integro-differential equation has been studied in [14] by means of Fourier series. One may also see [7], where the theory of analytic semigroups is used in terms of interpolation spaces to solve a boundary value problem in linear viscoelasticity. An abstract Volterra equation, as an abstract model for equations of linear viscoelasticity, has been studied in [5]. In a previous work [11], well-posedness and regularity of the problem (2.11), which is a simplified form (synchronous viscoelasticity) of the model problem (2.6), was studied in the framework of the semigroup of linear operators. The drawback of the framework is that this does not admit non-homogeneous Neumann boundary condition, while in practice mixed homogeneous Dirichlet and non-homogeneous Neumann boundary conditions are of special interest. Here we investigate existence, uniqueness and regularity of the solution of the problem (2.6) by means of the Galerkin approximation method.

There are also models with exponential kernels, smooth kernels, which describe polymeric materials, e.g., natural and synthetic rubber. Existence, uniqueness, and regularity of the solution of such models can be adapted from, e.g., [5]. The drawback with this kind of models is that it requires a large number of exponential kernels to describe the behavior of the materials. This is the reason for introducing kernels of Mittag-Leffler type or fractional operators. In [2] and [9] it is shown that the classical viscoelastic model based on exponential kernels can describe the same viscoelastic behaviour as the fractional model if the number of kernels tend to infinity. Here we extend the presented approach so that regularity of any order of the solution for the models with smooth kernels can be proved.

In the sequel, in §2 we describe the construction of the model problem (2.6) and we define a weak (generalized) solution. Then in §3, using Galerkin's method, we prove existence and uniqueness of the weak solution of the problem. Finally, in §4 we study regularity of the solution and limitations for higher regularity. We also show that higher regularity of any order of the solution of models with smooth kernels can be achieved.

2. FRACTIONAL ORDER VISCOELASTICITY

Let σ_{ij} , ϵ_{ij} and u_i denote, respectively, the usual stress tensor, strain tensor and displacement vector. We recall that the linear strain tensor is defined by,

$$\epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

With the decompositions

$$s_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij}, \quad e_{ij} = \epsilon_{ij} - \frac{1}{3} \epsilon_{kk} \delta_{ij},$$

the constitutive equations are formulated as, see [3],

$$(2.1) \quad \begin{aligned} s_{ij}(t) + \tau_1^{\alpha_1} D_t^{\alpha_1} s_{ij}(t) &= 2G_\infty e_{ij}(t) + 2G\tau_1^{\alpha_1} D_t^{\alpha_1} e_{ij}(t), \\ \sigma_{kk}(t) + \tau_2^{\alpha_2} D_t^{\alpha_2} \sigma_{kk}(t) &= 3K_\infty \epsilon_{kk}(t) + 3K\tau_2^{\alpha_2} D_t^{\alpha_2} \epsilon_{kk}(t), \end{aligned}$$

with initial conditions

$$s_{ij}(0+) = 2Ge_{ij}(0+), \quad \sigma_{kk}(0+) = 3K\epsilon_{kk}(0+),$$

meaning that the initial response follows Hooke's elastic law. Here G , K are the instantaneous (unrelaxed) modulus, and G_∞ , K_∞ are the long-time (relaxed) modulus. Note that we have two relaxation times, $\tau_1, \tau_2 > 0$, and fractional orders of differentiation, $\alpha_1, \alpha_2 \in (0, 1)$, where the fractional order derivative is defined by

$$D_t^\alpha f(t) = D_t D_t^{-(1-\alpha)} f(t) = D_t \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f(s) ds.$$

The constitutive equations (2.1) can be solved for σ by means of Laplace transformation, [9]:

$$\begin{aligned} s_{ij}(t) &= 2G \left(e_{ij}(t) - \frac{G - G_\infty}{G} \int_0^t \theta_1(t-s) e_{ij}(s) ds \right), \\ \sigma_{kk}(t) &= 3K \left(\epsilon_{kk}(t) - \frac{K - K_\infty}{K} \int_0^t \theta_2(t-s) \epsilon_{kk}(s) ds \right), \end{aligned}$$

where, for $i = 1, 2$,

$$\theta_i(t) = -\frac{d}{dt} E_{\alpha_i} \left(-\left(\frac{t}{\tau_i}\right)^{\alpha_i} \right), \quad E_{\alpha_i}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(1+n\alpha_i)},$$

and E_{α_i} is the Mittag-Leffler function of order α_i . Then we define parameters γ_i , and the Lamé constants μ , λ ,

$$\gamma_1 = \frac{G - G_\infty}{G}, \quad \gamma_2 = \frac{K - K_\infty}{K}, \quad \mu = G, \quad \lambda = K - \frac{2}{3}G.$$

We also define $\beta_i = \gamma_i \theta_i$, and the constitutive equations become

$$\begin{aligned} \sigma_{ij}(t) &= \left(2\mu \epsilon_{ij}(t) + \lambda \epsilon_{kk}(t) \delta_{ij} \right) - 2\mu \int_0^t \beta_1(t-s) \left(\epsilon_{ij}(s) - \frac{1}{3} \epsilon_{kk}(s) \delta_{ij} \right) ds \\ &\quad - \frac{3\lambda + 2\mu}{3} \int_0^t \beta_2(t-s) \epsilon_{kk}(s) \delta_{ij} ds. \end{aligned}$$

The kernels are weakly singular, for $i = 1, 2$:

$$\begin{aligned} (2.2) \quad \beta_i(t) &= -\gamma_i \frac{d}{dt} E_{\alpha_i} \left(-\left(\frac{t}{\tau_i}\right)^{\alpha_i} \right) = \gamma_i \frac{\alpha_i}{\tau_i} \left(\frac{t}{\tau_i}\right)^{-1+\alpha_i} E'_{\alpha_i} \left(-\left(\frac{t}{\tau_i}\right)^{\alpha_i} \right) \\ &\approx C t^{-1+\alpha_i}, \quad t \rightarrow 0, \end{aligned}$$

and we note the properties

$$\begin{aligned} (2.3) \quad \beta_i(t) &\geq 0, \\ \|\beta_i\|_{L_1(\mathbb{R}^+)} &= \int_0^\infty \beta_i(t) dt = \gamma_i (E_{\alpha_i}(0) - E_{\alpha_i}(\infty)) = \gamma_i < 1. \end{aligned}$$

The equations of motion are

$$\begin{aligned} (2.4) \quad \rho u_{i,tt} - \sigma_{ij,j} &= f_i, & \text{in } \Omega, \\ u_i &= 0, & \text{on } \Gamma_D, \\ \sigma_{ij} n_j &= g_i, & \text{on } \Gamma_N, \end{aligned}$$

where u is the displacement vector, ρ is the (constant) mass density, f and g represent, respectively, the volume and surface loads. We let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$,

be a bounded polygonal domain with boundary $\Gamma_D \cup \Gamma_N = \partial\Omega$, $\Gamma_D \cap \Gamma_N = \emptyset$ and $\text{meas}(\Gamma_D) \neq 0$. We set

$$(2.5) \quad \begin{aligned} (Au)_i &= -(2\mu\epsilon_{ij}(u) + \lambda\epsilon_{kk}(u)\delta_{ij})_{,j}, \\ (A_1u)_i &= -2\mu(\epsilon_{ij}(u) - \frac{1}{3}\epsilon_{kk}(u)\delta_{ij})_{,j}, \\ (A_2u)_i &= -\frac{3\lambda + 2\mu}{3}(\epsilon_{kk}(u)\delta_{ij})_{,j}, \end{aligned}$$

and clearly we have $A = A_1 + A_2$. Now, we write the equations of motion (2.4) in the strong form, (we denote time derivatives with $'\cdot'$),

$$(2.6) \quad \begin{aligned} \rho\ddot{u}(x, t) + Au(x, t) &= f(x, t) && \text{in } \Omega \times (0, T), \\ -\sum_{i=1}^2 \int_0^t \beta_i(t-s)A_iu(x, s) ds &= f(x, t) && \text{in } \Omega \times (0, T), \\ u(x, t) &= 0 && \text{on } \Gamma_D \times (0, T), \\ \sigma(u; x, t) \cdot n &= g(x, t) && \text{on } \Gamma_N \times (0, T), \\ u(x, 0) &= u^0(x) && \text{in } \Omega, \\ \dot{u}(x, 0) &= v^0(x) && \text{in } \Omega. \end{aligned}$$

We define the bilinear form (with the usual summation convention)

$$a(u, v) = \int_{\Omega} (2\mu\epsilon_{ij}(u)\epsilon_{ij}(v) + \lambda\epsilon_{ii}(u)\epsilon_{jj}(v)) dx, \quad \forall u, v \in V,$$

which is coercive, and in a similar way, corresponding to A_i , the bilinear forms $a_i(u, v)$, $i = 1, 2$. We introduce the function spaces $H = L_2(\Omega)^d$, $H_{\Gamma_N} = L_2(\Gamma_N)^d$, and $V = \{v \in H^1(\Omega)^d : v|_{\Gamma_D} = 0\}$. We denote the norms in H and H_{Γ_N} by $\|\cdot\|$ and $\|\cdot\|_{\Gamma_N}$, respectively, and we equip V with the inner product $a(\cdot, \cdot)$ and norm $\|v\|_V^2 = a(v, v)$. It can be shown that A , A_1 and A_2 are self-adjoint, positive definite linear operators (not bounded). We also note that, for $v \in V$,

$$(2.7) \quad a_i(v, v) \leq \|v\|_V^2.$$

Now we define a weak solution to be a function $u = u(x, t)$ that satisfies

$$(2.8) \quad u \in L_2((0, T); V), \quad \dot{u} \in L_2((0, T); H), \quad \ddot{u} \in L_2((0, T); V^*),$$

$$(2.9) \quad \begin{aligned} \rho\langle \ddot{u}(t), v \rangle + a(u(t), v) - \sum_{i=1}^2 \int_0^t \beta_i(t-s)a_i(u(s), v) ds \\ = (f(t), v) + (g(t), v)_{\Gamma_N}, \quad \forall v \in V, \text{ a.e. } t \in (0, T), \end{aligned}$$

$$(2.10) \quad u(0) = u^0, \quad \dot{u}(0) = v^0.$$

Here $(g(t), v)_{\Gamma_N} = \int_{\Gamma_N} g(t) \cdot v dS$, and $\langle \cdot, \cdot \rangle$ denotes the pairing of V^* and V . We note that (2.8) implies, by a classical result for Sobolev spaces, that $u \in \mathcal{C}([0, T]; H)$, $\dot{u} \in \mathcal{C}([0, T]; V^*)$ so that the initial conditions (2.10) make sense for $u^0 \in H$, $v^0 \in V^*$.

Remark 1. If we make the simplifying assumption (synchronous viscoelasticity):

$$\alpha = \alpha_1 = \alpha_2, \quad \tau = \tau_1 = \tau_2, \quad \theta = \theta_1 = \theta_2,$$

we may define $\gamma = \gamma_1 = \gamma_2$, so that $\beta = \beta_1 = \beta_2$. Then the equations of motion is

$$(2.11) \quad \rho\ddot{u}(x, t) + Au(x, t) - \int_0^t \beta(t-s)Au(x, s) ds = f(x, t) \quad \text{in } \Omega \times (0, T),$$

together with the boundary and initial conditions in (2.6).

Well-posedness and regularity of the simplified problem (2.11) was studied in [11], in the framework of the semigroup of linear operators. The drawback of the framework is that this does not admit non-homogeneous Neumann boundary condition.

3. EXISTENCE AND UNIQUENESS

In this section we prove existence and uniqueness of a weak solution of (2.6) using Galerkin's method, in a similar way for hyperbolic PDE's in [6], [10]. To this end, we first introduce the Galerkin approximation of a weak solution of (2.6) in a classical way, and we obtain a priori estimates for approximate solutions. These will be used to construct a weak solution, and then uniqueness will be verified.

We recall (2.2), (2.3) and we define the functions

$$\xi_i(t) = \gamma_i - \int_0^t \beta_i(s) ds = \int_t^\infty \beta_i(s) ds = \gamma_i E_{\alpha_i}(t), \quad i = 1, 2,$$

and it is easy to see that

$$(3.1) \quad D_t \xi_i(t) = -\beta_i(t) < 0, \quad \xi_i(0) = \gamma_i, \quad \lim_{t \rightarrow \infty} \xi_i(t) = 0, \quad 0 < \xi_i(t) \leq \gamma_i.$$

Besides, ξ_i are completely monotonic functions, that is,

$$(-1)^j D_t^j \xi_i(t) \geq 0, \quad t \in (0, \infty), \quad j \in \mathbb{N},$$

since the Mittag-Leffler functions E_{α_i} , $\alpha_i \in [0, 1]$ are completely monotonic. Consequently, an important property of ξ_i , $i = 1, 2$, is that, they are positive type kernels, that is, they are continuous and, for any $T \geq 0$, satisfy

$$(3.2) \quad \int_0^T \int_0^t \xi_i(t-s) \phi(t) \phi(s) ds dt \geq 0, \quad \forall \phi \in \mathcal{C}([0, T]).$$

3.1. Galerkin approximations. Let $\{(\lambda_j, \varphi_j)\}_{j=1}^\infty$ be the eigenpairs of the weak eigenvalue problem

$$(3.3) \quad a(\varphi, v) = \lambda(\varphi, v), \quad \forall v \in V.$$

It is known that $\{\varphi_j\}_{j=1}^\infty$ can be chosen to be an ON-basis in H and an orthogonal basis for V .

Now, for a fixed positive integer $m \in \mathbb{N}$, we seek a function of the form

$$(3.4) \quad u_m(t) = \sum_{j=1}^m d_j(t) \varphi_j$$

to satisfy

$$(3.5) \quad \begin{aligned} \rho(\ddot{u}_m(t), \varphi_k) + a(u_m(t), \varphi_k) - \sum_{i=1}^2 \int_0^t \beta_i(t-s) a_i(u_m(s), \varphi_k) ds \\ = (f(t), \varphi_k) + (g(t), \varphi_k)_{\Gamma_N}, \quad k = 1, \dots, m, \quad t \in (0, T), \end{aligned}$$

with initial conditions

$$(3.6) \quad u_m(0) = \sum_{j=1}^m (u^0, \varphi_j) \varphi_j, \quad \dot{u}_m(0) = \sum_{j=1}^m (v^0, \varphi_j) \varphi_j.$$

Theorem 1. For each $m \in \mathbb{N}$, there exists a unique function u_m of the form (3.4) satisfying (3.5)-(3.6). Moreover, if $u^0 \in V$, $v^0 \in H$, $g \in W_1^1((0, T); H_{\Gamma_N})$, $f \in L_2((0, T); H)$, there is a constant $C = C(\Omega, \gamma_1, \gamma_2, \rho, T)$ such that,

$$(3.7) \quad \begin{aligned} & \|u_m\|_{L_\infty((0, T); V)} + \|\dot{u}_m\|_{L_\infty((0, T); H)} + \|\ddot{u}_m\|_{L_2((0, T); V^*)} \\ & \leq C\{\|u^0\|_V + \|v^0\| + \|g\|_{W_1^1((0, T); H_{\Gamma_N})} + \|f\|_{L_2((0, T); H)}\}. \end{aligned}$$

Proof. Using (3.4) and the fact that $\{\varphi_j\}_{j=1}^\infty$ is an ON-basis for H and a solution of the eigenvalue problem (3.3), we obtain from (3.5) that,

$$(3.8) \quad \begin{aligned} & \rho \ddot{d}_k(t) + \lambda_k d_k(t) - \sum_{j=1}^m \sum_{i=1}^2 a_i(\varphi_j, \varphi_k)(\beta_i * d_j)(t) \\ & = f_k(t) + g_k(t), \quad k = 1, \dots, m, t \in (0, T), \end{aligned}$$

where $*$ denotes the convolution, and $f_k(t) = (f(t), \varphi_k)$, $g_k(t) = (g(t), \varphi_k)_{\Gamma_N}$. This is a linear system of second order ODE's with initial conditions

$$(3.9) \quad d_k(0) = (u^0, \varphi_k), \quad \dot{d}_k(0) = (v^0, \varphi_k), \quad k = 1, \dots, m.$$

The Laplace transform can be used, for example, to find the unique solution of the system.

Taking the Laplace transform of (3.8) we get (we use an over-hat for the Laplace transform),

$$(3.10) \quad \begin{aligned} & (s^2 \rho + \lambda_k) \hat{d}_k(s) - \sum_{j=1}^m \sum_{i=1}^2 a_i(\varphi_j, \varphi_k) \hat{\beta}_i(s) \hat{d}_j(s) \\ & = \hat{f}_k(s) + \hat{g}_k(s) + s \rho d_k(0) + \rho \dot{d}_k(0), \\ & k = 1, \dots, m, \quad s > \frac{1}{\min\{\tau_1, \tau_2\}}. \end{aligned}$$

Now we show that there is a unique solution for this system, that is, the inverse Laplace transform is computable. This proves that the linear system (3.8) with initial conditions (3.9) has a unique solution, that will prove the first part of the theorem.

Indeed, the Laplace transform of the Mittag-Leffler function is

$$\mathcal{L}(E_\alpha(at^\alpha)) = \frac{s^{\alpha-1}}{s^\alpha - a}, \quad s > |a|^{1/\alpha}.$$

Hence for the kernels β_i , $i = 1, 2$, defined in (2.2) we have

$$\begin{aligned} \hat{\beta}_i(s) &= \mathcal{L}(\beta_i(t)) = -\gamma_i s \mathcal{L}(E_{\alpha_i}(-\tau_i^{-\alpha_i} t^{\alpha_i})) + \gamma_i E_{\alpha_i}(0) \\ &= -\gamma_i s \frac{s^{\alpha_i-1}}{s^{\alpha_i} + \tau_i^{-\alpha_i}} + \gamma_i = \gamma_i - \gamma_i \frac{s^{\alpha_i}}{s^{\alpha_i} + \tau_i^{-\alpha_i}} \\ &= \frac{\gamma_i}{(\tau_i s)^{\alpha_i} + 1} < 1, \quad s > \tau_i^{-1}. \end{aligned}$$

Since $\hat{\beta}_i$ are decreasing functions, at most at one point, say $s = \tilde{s} > 1/\min\{\tau_1, \tau_2\}$, we might have $\hat{\beta}_1(s) = \hat{\beta}_2(s)$. Therefore without loss of generality we can assume $\hat{\beta}_1 \geq \hat{\beta}_2$ for $s \geq \tilde{s}$. Then equation (3.10), using $a(\cdot, \cdot) = a_1(\cdot, \cdot) + a_2(\cdot, \cdot)$, can be

written in the form

$$(3.11) \quad \begin{aligned} (\mathbf{C}_1(s) + (\hat{\beta}_1(s) - \hat{\beta}_2(s))\mathbf{A}_2)\hat{\mathbf{D}}(s) &= \hat{\mathbf{F}}(s), & \text{if } s \geq \tilde{s}, \\ (\mathbf{C}_2(s) + (\hat{\beta}_2(s) - \hat{\beta}_1(s))\mathbf{A}_1)\hat{\mathbf{D}}(s) &= \hat{\mathbf{F}}(s), & \text{if } \frac{1}{\min\{\tau_1, \tau_2\}} < s < \tilde{s}, \end{aligned}$$

where

$$\begin{aligned} \hat{\mathbf{D}}(s) &= (\hat{d}_k(s))_{k=1}^m, & \hat{\mathbf{F}}(s) &= (\hat{f}_k(s) + \hat{g}_k(s) + s\rho d_k(0) + \rho \dot{d}_k(0))_{k=1}^m, \\ \mathbf{A}_i &= (a_i(\varphi_j, \varphi_k))_{j,k=1}^m, & \mathbf{C}_i(s) &= \text{diag}\{(s^2\rho + \lambda_k - \lambda_k\hat{\beta}_i(s))_{k=1}^m\}, \quad i = 1, 2. \end{aligned}$$

Obviously \mathbf{C}_i and \mathbf{A}_i , $i = 1, 2$ are symmetric positive definite matrices, and therefore the coefficient matrices of the linear systems above are symmetric positive definite. That is, they are uniquely solvable and this proves the first part of the theorem.

Now we prove the a priori estimate (3.7). Since $\beta_i(t-s) = D_s \xi_i(t-s)$ and $\xi_i(0) = \gamma_i$, by (3.1), we can write (3.5), after partial integration in time, as

$$\begin{aligned} &\rho(\ddot{u}_m(t), \varphi_k) + a(u_m(t), \varphi_k) - \sum_{i=1}^2 \gamma_i a_i(u_m(t), \varphi_k) \\ &\quad + \sum_{i=1}^2 \int_0^t \xi_i(t-s) a_i(\dot{u}_m(s), \varphi_k) ds \\ &= (f(t), \varphi_k) + (g(t), \varphi_k)_{\Gamma_N} \\ &\quad - \sum_{i=1}^2 \xi_i(t) a_i(u_m(0), \varphi_k), \quad k = 1, \dots, m, t \in (0, T). \end{aligned}$$

Then multiplying by $\dot{d}_k(t)$, summing over $k = 1, \dots, m$, and integrating with respect to t , we have

$$\begin{aligned} &\rho \|\dot{u}_m(t)\|^2 + (1 - \bar{\gamma}) \|u_m(t)\|_V^2 + 2 \sum_{i=1}^2 \int_0^t \int_0^r \xi_i(r-s) a_i(\dot{u}_m(s), \dot{u}_m(r)) ds dr \\ &\leq \rho \|\dot{u}_m(0)\|^2 + (1 - \underline{\gamma}) \|u_m(0)\|_V^2 \\ &\quad + 2 \int_0^t (f(r), \dot{u}_m(r)) dr + 2 \int_0^t (g(r), \dot{u}_m(r))_{\Gamma_N} dr \\ &\quad - 2 \sum_{i=1}^2 \int_0^t \xi_i(r) a_i(u_m(0), \dot{u}_m(r)) dr, \end{aligned}$$

where $\bar{\gamma} = \max\{\gamma_1, \gamma_2\}$ and $\underline{\gamma} = \min\{\gamma_1, \gamma_2\}$. We note that $0 < \underline{\gamma}, \bar{\gamma} < 1$. Since ξ_i , $i = 1, 2$ are positive type kernels, recalling (3.2), the third term of the left hand side is non-negative. Then integration by parts in the last two terms at the right

side yields

$$\begin{aligned}
& \rho \|\dot{u}_m(t)\|^2 + (1 - \bar{\gamma}) \|u_m(t)\|_V^2 \\
& \leq \rho \|\dot{u}_m(0)\|^2 + (1 - \underline{\gamma}) \|u_m(0)\|_V^2 + 2 \int_0^t (f(r), \dot{u}_m(r)) dr \\
& \quad - 2 \int_0^t (\dot{g}(r), u_m(r))_{\Gamma_N} dr + 2(g(t), u_m(t))_{\Gamma_N} - 2(g(0), u_m(0))_{\Gamma_N} \\
& \quad - 2 \sum_{i=1}^2 \int_0^t \beta_i(r) a_i(u_m(0), u_m(r)) dr \\
& \quad - 2 \sum_{i=1}^2 \xi_i(t) a_i(u_m(0), u_m(t)) + 2 \sum_{i=1}^2 \xi_i(0) a_i(u_m(0), u_m(0)).
\end{aligned}$$

This, using the Cauchy-Schwarz inequality, the trace theorem, $\|\beta_i\|_{L_1(\mathbb{R}^+)} = \gamma_i$, $\xi_i(t) \leq \xi_i(0) = \gamma_i$, and (2.7), implies

$$\begin{aligned}
& \rho \|\dot{u}_m(t)\|^2 + (1 - \bar{\gamma}) \|u_m(t)\|_V^2 \\
& \leq \rho \|\dot{u}_m(0)\|^2 + (1 - \underline{\gamma}) \|u_m(0)\|_V^2 \\
& \quad + 2/C_1 \max_{0 \leq r \leq t} \|\dot{u}_m(r)\|^2 + C_1 \left(\int_0^t \|f(r)\| dr \right)^2 \\
& \quad + 2C_{\text{Trace}}/C_2 \max_{0 \leq r \leq t} \|u_m(r)\|_V^2 + 2C_{\text{Trace}}C_2 \left(\int_0^t \|\dot{g}(r)\|_{H_{\Gamma_N}} \right)^2 \\
& \quad + 2C_{\text{Trace}}/C_3 \|u_m(t)\|_V^2 + 2C_{\text{Trace}}C_3 \|g(t)\|_{H_{\Gamma_N}}^2 \\
& \quad + 2C_{\text{Trace}}/C_4 \|u_m(0)\|_V^2 + 2C_{\text{Trace}}C_4 \|g(0)\|_{H_{\Gamma_N}}^2 \\
& \quad + 2/C_5 \left(\sum_{i=1}^2 \gamma_i \right) \|u_m(0)\|_V^2 + 2C_5 \left(\sum_{i=1}^2 \gamma_i \right) \max_{0 \leq r \leq t} \|u_m(r)\|_V^2 \\
& \quad + 2/C_6 \left(\sum_{i=1}^2 \gamma_i \right) \|u_m(0)\|_V^2 + 2C_6 \left(\sum_{i=1}^2 \gamma_i \right) \|u_m(t)\|_V^2 + 2 \left(\sum_{i=1}^2 \gamma_i \right) \|u_m(0)\|_V^2.
\end{aligned}$$

Hence, considering the facts that $C_{\text{Trace}} = C(\Omega)$, $\|\dot{u}_m(0)\| \leq \|v^0\|$, and $\|u_m(0)\|_V \leq \|u^0\|_V$, for some constant $C = C(\Omega, \gamma_1, \gamma_2, \rho, T)$, we have

$$\begin{aligned}
& \|\dot{u}_m\|_{L_\infty((0,T);H)}^2 + \|u_m\|_{L_\infty((0,T);V)}^2 \\
& \leq C \{ \|v^0\|^2 + \|u^0\|_V^2 + \|g\|_{L_\infty((0,T);H_{\Gamma_N})}^2 \\
& \quad + \|f\|_{L_1((0,T);H)}^2 + \|\dot{g}\|_{L_1((0,T);H_{\Gamma_N})}^2 \}.
\end{aligned}$$

This, and the facts that $\|g\|_{L_\infty((0,T);H_{\Gamma_N})} \leq C\|g\|_{W_1^1((0,T);H_{\Gamma_N})}$, by Sobolev inequality, and $\|f\|_{L_1((0,T);H)} \leq C\|f\|_{L_2((0,T);H)}$, imply

$$\begin{aligned}
(3.12) \quad & \|\dot{u}_m\|_{L_\infty((0,T);H)}^2 + \|u_m\|_{L_\infty((0,T);V)}^2 \\
& \leq C \{ \|v^0\|^2 + \|u^0\|_V^2 + \|g\|_{W_1^1((0,T);H_{\Gamma_N})}^2 + \|f\|_{L_2((0,T);H)}^2 \}.
\end{aligned}$$

Now we need to find a bound for \ddot{u}_m . For any fixed $v \in V$ with $\|v\|_V \leq 1$, we write $v = v^1 + v^2$, where $v^1 \in \text{span}\{\varphi_j\}_{j=1}^m$, $v^2 \in \text{span}(\{\varphi_j\}_{j=1}^m)^\perp$. We note that

$\|v^1\|_V \leq 1$. Then from (3.5) we obtain,

$$\begin{aligned} \rho \langle \ddot{u}_m(t), v \rangle &= \rho \langle \ddot{u}_m(t), v^1 \rangle = (f(t), v^1) + (g(t), v^1)_{\Gamma_N} - a(u_m(t), v^1) \\ &\quad + \sum_{i=1}^2 \int_0^t \beta_i(t-s) a_i(u_m(s), v^1) ds, \end{aligned}$$

that, using the Cauchy-Schwarz inequality, the trace theorem, and (2.7), implies

$$|\langle \ddot{u}_m(t), v \rangle| \leq \frac{1}{\rho} \left(\|f(t)\| + C_{\text{Trace}} \|g(t)\| + \|u_m(t)\|_V + \max_{0 \leq s \leq t} \|u_m(s)\|_V \sum_{i=1}^2 \gamma_i \right).$$

This, using (3.12), in a standard way implies

$$\begin{aligned} \|\ddot{u}_m\|_{L_2((0,T);V^*)}^2 &\leq C \{ \|f\|_{L_2((0,T);H)}^2 + \|g\|_{L_2((0,T);H_{\Gamma_N})}^2 \\ &\quad + \|v^0\|^2 + \|u^0\|_V^2 + \|g\|_{W_1^1((0,T);H_{\Gamma_N})}^2 \}. \end{aligned}$$

Therefore, for some constant $C = C(\Omega, \gamma_1, \gamma_2, \rho, T)$,

$$\|\ddot{u}_m\|_{L_2((0,T);V^*)}^2 \leq C \{ \|v^0\|^2 + \|u^0\|_V^2 + \|g\|_{W_1^1((0,T);H_{\Gamma_N})}^2 + \|f\|_{L_2((0,T);H)}^2 \}.$$

This and (3.12) imply the estimate (3.7), and the proof is complete. \square

Remark 2. For the simplified model problem (2.11) in Remark 1, since $\hat{\beta}_1 = \hat{\beta}_2$, the linear system (3.11) is reduced to

$$\mathbf{C}\hat{\mathbf{D}} = \hat{\mathbf{F}},$$

where $\mathbf{C} = \mathbf{C}_1 = \mathbf{C}_2$. This system has a unique solution, since \mathbf{C} is symmetric positive definite. Hence there exists a unique function u_m of the form (3.4) satisfying

$$\begin{aligned} \rho \langle \ddot{u}_m(t), \varphi_k \rangle + a(u_m(t), \varphi_k) - \int_0^t \beta(t-s) a(u_m(s), \varphi_k) ds \\ = (f(t), \varphi_k) + (g(t), \varphi_k)_{\Gamma_N}, \quad k = 1, \dots, m, t \in (0, T), \end{aligned}$$

with initial conditions (3.6). Moreover, the a priori estimate (3.7) still holds with $C = C(\Omega, \gamma, \rho, T)$.

3.2. Existence and uniqueness of the weak solution. Now, we use Theorem 1 to prove existence and uniqueness of the weak solution of (2.6), that is a solution of (2.8)–(2.10).

Theorem 2. *If $u^0 \in V$, $v^0 \in H$, $g \in W_1^1((0, T); H_{\Gamma_N})$, $f \in L_2((0, T); H)$, there exists a unique weak solution of (2.6).*

Proof. 1. We note that the estimate (3.7) does not depend on m , so we have

$$\begin{aligned} \|u_m\|_{L_\infty((0,T);V)} + \|\dot{u}_m\|_{L_\infty((0,T);H)} + \|\ddot{u}_m\|_{L_2((0,T);V^*)} \\ \leq K = K(\Omega, \gamma_1, \gamma_2, T, u^0, v^0, f, g). \end{aligned}$$

That is,

$$\begin{aligned} (3.13) \quad \{u_m\}_1^\infty &\text{ is bounded in } L_\infty((0, T); V) \subset L_2((0, T); V), \\ \{\dot{u}_m\}_1^\infty &\text{ is bounded in } L_\infty((0, T); H) \subset L_2((0, T); H), \\ \{\ddot{u}_m\}_1^\infty &\text{ is bounded in } L_2((0, T); V^*). \end{aligned}$$

2. First we prove existence. From (3.13) and a classical result in functional analysis, we conclude that the sequencess $\{u_m\}_{m=1}^\infty$, $\{\dot{u}_m\}_{m=1}^\infty$, $\{\ddot{u}_m\}_{m=1}^\infty$ are weakly

precompact. That is, there are subsequences of $\{u_m\}_{m=1}^\infty$, $\{\dot{u}_m\}_{m=1}^\infty$, $\{\ddot{u}_m\}_{m=1}^\infty$, such that

$$(3.14) \quad \begin{aligned} u_l &\rightharpoonup u && \text{in } L_2((0, T); V), \\ \dot{u}_l &\rightharpoonup \dot{u} && \text{in } L_2((0, T); H), \\ \ddot{u}_l &\rightharpoonup \ddot{u} && \text{in } L_2((0, T); V^*), \end{aligned}$$

where the index l is a replacement of the label of the subsequences and ' \rightharpoonup ' denotes weak convergence. Consequently, (2.8) holds true and we need to verify (2.9) and (2.10). To show (2.9) we fix a positive integer N and we choose $v \in \mathcal{C}([0, T]; V)$ of the form

$$(3.15) \quad v(t) = \sum_{j=1}^N h_j(t) \varphi_j.$$

Then we take $l \geq N$ and by (3.5) we have

$$(3.16) \quad \begin{aligned} \int_0^T \left(\rho \langle \ddot{u}_l, v \rangle + a(u_l, v) - \sum_{i=1}^2 \int_0^t \beta_i(t-s) a_i(u_l(s), v) ds \right) dt \\ = \int_0^T ((f, v) + (g, v)_{\Gamma_N}) dt. \end{aligned}$$

This, by (3.14), implies in the limit

$$(3.17) \quad \begin{aligned} \int_0^T \left(\rho \langle \ddot{u}, v \rangle + a(u, v) - \sum_{i=1}^2 \int_0^t \beta_i(t-s) a_i(u(s), v) ds \right) dt \\ = \int_0^T ((f, v) + (g, v)_{\Gamma_N}) dt. \end{aligned}$$

Since functions of the form (3.15) are dense in $L_2((0, T); V)$, this equality then holds for all functions $v \in L_2((0, T); V)$, and further it implies (2.9).

Now, we need to show that u satisfies the initial conditions (2.10). Let $v \in \mathcal{C}^2([0, T]; V)$ be any function with $v(T) = \dot{v}(T) = 0$. Then by partial integration in (3.16) we have

$$\begin{aligned} \int_0^T \left(\rho \langle u_l, \ddot{v} \rangle + a(u_l, v) - \sum_{i=1}^2 \int_0^t \beta_i(t-s) a_i(u_l(s), v) ds \right) dt \\ = \int_0^T ((f, v) + (g, v)_{\Gamma_N}) dt - \rho(u_l(0), \dot{v}(0)) + \rho(\dot{u}_l(0), v(0)), \end{aligned}$$

so that, recalling (3.14) and (3.6), in the limit we conclude,

$$\begin{aligned} \int_0^T \left(\rho \langle u, \ddot{v} \rangle + a(u, v) - \sum_{i=1}^2 \int_0^t \beta_i(t-s) a_i(u(s), v) ds \right) dt \\ = \int_0^T ((f, v) + (g, v)_{\Gamma_N}) dt - \rho(u^0, \dot{v}(0)) + \rho(\dot{v}^0, v(0)). \end{aligned}$$

On the other hand integration by parts in (3.17) gives,

$$\begin{aligned} & \int_0^T \left(\rho \langle u, \ddot{v} \rangle + a(u, v) - \sum_{i=1}^2 \int_0^t \beta_i(t-s) a_i(u(s), v) ds \right) dt \\ &= \int_0^T ((f, v) + (g, v)_{\Gamma_N}) dt - \rho(u(0), \dot{v}(0)) + \rho(v(0), v(0)). \end{aligned}$$

Comparing the last two identities we conclude (2.10), since $v(0)$, $\dot{v}(0)$ are arbitrary. Hence u is a weak solution of (2.6).

3. It remains to prove uniqueness. To this end, we show that $u = 0$ is the solution of (2.8)–(2.10) for $u^0 = v^0 = f = g = 0$. Let us fix $r \in [0, T]$ and define

$$v(t) = \begin{cases} \int_t^r u(\omega) d\omega & 0 \leq t \leq r \\ 0 & r \leq t \leq T. \end{cases}$$

We note that

$$(3.18) \quad v(t) \in V, \quad v(r) = 0, \quad \dot{v}(t) = -u(t).$$

Then inserting v in (2.9) and integrating with respect to t , we have

$$(3.19) \quad \int_0^r (\rho \langle \ddot{u}, v \rangle + a(u, v)) dt - \sum_{i=1}^2 \int_0^r \int_0^t \beta_i(t-s) a_i(u(s), v(t)) ds dt = 0.$$

For the second term, recalling $-\beta_i(t) = D_t \xi_i(t)$, $i = 1, 2$ from (3.1), we obtain

$$\begin{aligned} - \int_0^r \int_0^t \beta_i(t-s) a_i(u(s), v(t)) ds dt &= \int_0^r \int_s^r D_t \xi_i(t-s) a_i(u(s), v(t)) dt ds \\ &= \int_0^r \xi_i(r-s) a_i(u(s), v(r)) ds \\ &\quad - \int_0^r \xi_i(0) a_i(u(s), v(s)) ds \\ &\quad - \int_0^r \int_s^r \xi_i(t-s) a_i(u(s), \dot{v}(t)) dt ds \\ &= -\gamma_i \int_0^r a_i(u(s), v(s)) ds \\ &\quad + \int_0^r \int_0^t \xi_i(t-s) a_i(u(s), u(t)) ds dt, \end{aligned}$$

where we changed the order of integrals and we used integration by parts, $\xi_i(0) = \gamma_i$ from (3.1), and $v(r) = 0$ from (3.18). Therefore integration by parts in the first term of (3.19) yields

$$\begin{aligned} & \int_0^r (-\rho \langle \dot{u}, \dot{v} \rangle + a(u, v)) dt - \sum_{i=1}^2 \gamma_i \int_0^r a_i(u, v) dt \\ & \quad + \sum_{i=1}^2 \int_0^r \int_0^t \xi_i(t-s) a_i(u(s), u(t)) ds dt = 0. \end{aligned}$$

This, using (3.18), implies

$$\begin{aligned} \rho \|u(r)\|^2 - \rho \|u(0)\|^2 - \|v(r)\|_V^2 + \|v(0)\|_V^2 + \sum_{i=1}^2 \gamma_i \left(a_i(v(r), v(r)) - a_i(v(0), v(0)) \right) \\ + 2 \sum_{i=1}^2 \int_0^r \int_0^t \xi_i(t-s) a_i(u(s), u(t)) ds dt = 0. \end{aligned}$$

Consequently, recalling (3.2), $v(r) = 0$, $u(0) = 0$, $0 < \bar{\gamma} = \max\{\gamma_1, \gamma_2\} < 1$, and the fact that A_i are positive definite, we have

$$\rho \|u(r)\|^2 + (1 - \bar{\gamma}) \|v(0)\|_V^2 \leq 0,$$

that implies $u = 0$ a.e., and this completes the proof. \square

Remark 3. Theorem 2 also holds for the simplified problem (2.11), see Remark 1 and Remark 2. That is, with the assumptions in Theorem 2, there exists a unique weak solution for the simplified problem.

4. REGULARITY

Here we study the regularity of the unique weak solution of (2.6), that is, a solution of (2.8)–(2.10). We explain the limitations for higher regularity in Remark 4. We also prove higher regularity of any order of the solution of models with smooth kernels in Theorem 4.

Corollary 1. *If $u^0 \in V$, $v^0 \in H$, $g \in W_1^1((0, T); H_{\Gamma_N})$, and $f \in L_2((0, T); H)$, then for the unique solution u of (2.8)–(2.10) we have*

$$(4.1) \quad u \in L_\infty((0, T); V), \quad \dot{u} \in L_\infty((0, T); H), \quad \ddot{u} \in L_2((0, T); V^*).$$

Moreover we have the estimate

$$(4.2) \quad \|u\|_{L_\infty((0, T); V)} + \|\dot{u}\|_{L_\infty((0, T); H)} + \|\ddot{u}\|_{L_2((0, T); V^*)} \\ \leq C \{ \|u^0\|_V + \|v^0\| + \|g\|_{W_1^1((0, T); H_{\Gamma_N})} + \|f\|_{L_2((0, T); H)} \}.$$

Proof. It is known that if $u_m \rightharpoonup u$, then

$$\|u\| \leq \liminf_{m \rightarrow \infty} \|u_m\|.$$

Then, by (3.14) and the a priori estimates (3.7), we conclude (4.1) and (4.2). \square

We note that, using Remark 3, Corollary 1 applies also to the simplified problem (2.11).

It is known from the theory of the elliptic operators, that global higher spatial regularity can not be obtained with mixed boundary conditions. Therefore we specialize to the homogeneous Dirichlet boundary condition, that is $\Gamma_N = \emptyset$, and assume that the polygonal domain Ω is convex. We recall the usual Sobolev spaces $H^r = H^r(\Omega)$ and we note that here $V = H_0^1(\Omega)$. We then use the extension of the operator A to an abstract operator with $\mathcal{D}(A) = H^2(\Omega)^d \cap V$ such that $a(u, v) = (Au, v)$ for sufficiently smooth u, v . We note that, the elliptic regularity holds, that is,

$$(4.3) \quad \|u\|_{H^2} \leq C \|Au\|, \quad u \in H^2(\Omega)^d \cap V.$$

Theorem 3. *We assume that $\Gamma_N = \emptyset$, and*

$$(4.4) \quad \sum_{i=1}^2 \int_0^t \beta_i(s) \, ds < 1 \text{ or } \int_0^t \max_{i=1,2} \beta_i(s) \, ds < \frac{1}{2}.$$

If $u^0 \in H^2$, $v^0 \in V$, and $\dot{f} \in L_2((0, T); H)$, then for the unique solution u of (2.8)-(2.10) we have

$$(4.5) \quad \begin{aligned} u &\in L_\infty((0, T); H^2), \quad \dot{u} \in L_\infty((0, T); V), \\ \ddot{u} &\in L_\infty((0, T); H), \quad \ddot{u} \in L_2((0, T); V^*). \end{aligned}$$

Moreover we have the estimate

$$(4.6) \quad \begin{aligned} &\|u\|_{L_\infty((0, T); H^2)} + \|\dot{u}\|_{L_\infty((0, T); V)} + \|\ddot{u}\|_{L_\infty((0, T); H)} + \|\ddot{u}\|_{L_2((0, T); V^*)} \\ &\leq C \{ \|u^0\|_{H^2} + \|v^0\|_V + \|f\|_{H^1((0, T); H)} \}. \end{aligned}$$

Proof. Differentiating (3.5) with respect to time, with notation $\underline{v} = \dot{v}$, we have

$$(4.7) \quad \begin{aligned} &\rho(\ddot{\underline{u}}_m(t), \varphi_k) + a(\underline{u}_m(t), \varphi_k) - \sum_{i=1}^2 \int_0^t \beta_i(t-s) a_i(\underline{u}_m(s), \varphi_k) \, ds \\ &= (\underline{f}(t), \varphi_k) + \sum_{i=1}^2 \beta_i(t) a_i(u_m(0), \varphi_k), \quad k = 1, \dots, m, t \in (0, T), \end{aligned}$$

with the initial conditions

$$(4.8) \quad \begin{aligned} \underline{u}_m(0) &= \dot{u}_m(0) = \sum_{j=1}^m (v^0, \varphi_j) \varphi_j, \\ \dot{\underline{u}}_m(0) &= \ddot{u}_m(0) = \sum_{j=1}^m (f(0) - Au_m(0), \varphi_j) \varphi_j. \end{aligned}$$

Then, using $\beta_i(t-s) = D_s \xi_i(t-s)$ from (3.1) and partial integration in time, we have

$$\begin{aligned} &\rho(\ddot{\underline{u}}_m(t), \varphi_k) + a(\underline{u}_m(t), \varphi_k) - \sum_{i=1}^2 \gamma_i a_i(\underline{u}_m(t), \varphi_k) \\ &+ \sum_{i=1}^2 \int_0^t \xi_i(t-s) a_i(\dot{\underline{u}}_m(s), \varphi_k) \, ds \\ &= (\underline{f}(t), \varphi_k) + \sum_{i=1}^2 \beta_i(t) a_i(u_m(0), \varphi_k) \\ &- \sum_{i=1}^2 \xi_i(t) a(\underline{u}_m(0), \varphi_k), \quad k = 1, \dots, m, t \in (0, T). \end{aligned}$$

Now, multiplying by $\ddot{d}_k(t)$, summing $k = 1, \dots, m$, and integration with respect to t , we have

$$\begin{aligned} & \rho \|\dot{\underline{u}}_m(t)\|^2 + (1 - \bar{\gamma}) \|\underline{u}_m(t)\|_V^2 \\ & + 2 \sum_{i=1}^2 \int_0^t \int_0^r \xi_i(r-s) a_i(\dot{\underline{u}}_m(s), \dot{\underline{u}}_m(r)) ds dr \\ & \leq \rho \|\dot{\underline{u}}_m(0)\|^2 + (1 - \underline{\gamma}) \|\underline{u}_m(0)\|_V^2 \\ & + 2 \int_0^t (\underline{f}(r), \dot{\underline{u}}_m(r)) dr + 2 \sum_{i=1}^2 \int_0^t \beta_i(r) a_i(u_m(0), \dot{\underline{u}}_m(r)) dr \\ & - 2 \sum_{i=1}^2 \int_0^t \xi_i(r) a_i(\underline{u}_m(0), \dot{\underline{u}}_m(r)) dr, \end{aligned}$$

where we recall that $\xi_i(0) = \gamma_i$, $\bar{\gamma} = \max\{\gamma_1, \gamma_2\}$, and $\underline{\gamma} = \min\{\gamma_1, \gamma_2\}$. Then, recalling the fact that ξ_i are positive definite (3.2) and integration by parts in the last term, we obtain

$$\begin{aligned} & \rho \|\dot{\underline{u}}_m(t)\|^2 + (1 - \bar{\gamma}) \|\underline{u}_m(t)\|_V^2 \\ & \leq \rho \|\dot{\underline{u}}_m(0)\|^2 + (1 - \underline{\gamma}) \|\underline{u}_m(0)\|_V^2 \\ & + 2 \int_0^t (\underline{f}(r), \dot{\underline{u}}_m(r)) dr + 2 \sum_{i=1}^2 \int_0^t \beta_i(r) a_i(u_m(0), \dot{\underline{u}}_m(r)) dr \\ & - 2 \sum_{i=1}^2 \int_0^t \beta_i(r) a_i(\underline{u}_m(0), \underline{u}_m(r)) dr \\ & - 2 \sum_{i=1}^2 \xi_i(t) a_i(\underline{u}_m(0), \underline{u}_m(t)) + 2 \sum_{i=1}^2 \xi_i(0) a_i(\underline{u}_m(0), \underline{u}_m(0)), \end{aligned}$$

that, using the Cauchy-Schwarz inequality, $\|\beta_i\|_{L_1(\mathbb{R}^+)} = \gamma_i$, $\xi_i(t) \leq \xi_i(0) = \gamma_i$, and (2.7), implies

$$\begin{aligned} & \rho \|\dot{\underline{u}}_m(t)\|^2 + (1 - \bar{\gamma}) \|\underline{u}_m(t)\|_V^2 \\ & \leq \rho \|\dot{\underline{u}}_m(0)\|^2 + (1 - \underline{\gamma}) \|\underline{u}_m(0)\|_V^2 \\ & + 2/C_1 \max_{0 \leq r \leq t} \|\dot{\underline{u}}_m(r)\|^2 + C_1 \left(\int_0^t \|\underline{f}(r)\| dr \right)^2 \\ & + 2/C_2 \left(\sum_{i=1}^2 \gamma_i \right) \|u_m(0)\|_{H^2}^2 + 2 \left(\sum_{i=1}^2 \gamma_i \right) C_2 \max_{0 \leq r \leq t} \|\dot{\underline{u}}_m(r)\|^2 \\ & + 2/C_3 \left(\sum_{i=1}^2 \gamma_i \right) \|\underline{u}_m(0)\|_V^2 + 2 \left(\sum_{i=1}^2 \gamma_i \right) C_3 \max_{0 \leq r \leq t} \|\underline{u}_m(r)\|_V^2 \\ & + 2/C_4 \left(\sum_{i=1}^2 \gamma_i \right) \|\underline{u}_m(0)\|_V^2 + 2 \left(\sum_{i=1}^2 \gamma_i \right) C_4 \|\underline{u}_m(t)\|_V^2 + 2 \left(\sum_{i=1}^2 \gamma_i \right) \|\underline{u}_m(0)\|_V^2. \end{aligned}$$

This implies, for some constant $C = C(\gamma_1, \gamma_2, \rho, T)$,

$$\begin{aligned} & \|\dot{\underline{u}}_m\|_{L_\infty((0,T);H)}^2 + \|\underline{u}_m\|_{L_\infty((0,T);V)}^2 \\ & \leq C \{ \|\dot{\underline{u}}_m(0)\|^2 + \|\underline{u}_m(0)\|_V^2 + \|u_m(0)\|_{H^2}^2 + \|\underline{f}\|_{L_1((0,T);H)}^2 \}. \end{aligned}$$

Then recalling $\underline{u} = \dot{u}$, the initial data from (4.8), and using

$$\|u_m(0)\|_{H^2} \leq \|u^0\|_{H^2}, \quad \|\dot{u}_m(0)\|_V \leq \|v^0\|_V,$$

we have

$$(4.9) \quad \begin{aligned} & \|\ddot{u}_m\|_{L_\infty((0,T);H)}^2 + \|\dot{u}_m\|_{L_\infty((0,T);V)}^2 \\ & \leq C\{\|u^0\|_{H^2}^2 + \|v^0\|_V^2 + \|f(0)\|^2 + \|\underline{f}\|_{L_1((0,T);H)}^2\}. \end{aligned}$$

We now find a bound for $\|u_m(t)\|_{H^2}$. We recall the eigenvalue problem (3.3) with eigenpairs $\{(\lambda_j, \varphi_j)\}_{j=1}^\infty$. Then we multiply (3.5) by $\lambda_k d_k(t)$ and add for $k = 1, \dots, m$ to obtain

$$(4.10) \quad a(u_m, Au_m) = (f - \rho \ddot{u}_m, Au_m) + \sum_{i=1}^2 \int_0^t \beta_i(t-s) a_i(u_m(s), Au_m(t)) ds.$$

This, using the Cauchy-Schwarz inequality and (2.7), implies

$$(4.11) \quad \begin{aligned} \|Au_m(t)\|^2 & \leq \frac{2}{\epsilon} \left(\|f(t)\|^2 + \rho^2 \|\ddot{u}_m(t)\|^2 \right) + \epsilon \|Au_m(t)\|^2 \\ & + \left(\sum_{i=1}^2 \int_0^t \beta_i(s) ds \right) \max_{0 \leq s \leq t} \|Au_m(s)\|^2, \end{aligned}$$

that, by elliptic regularity (4.3) and assumption (4.4), gives us

$$\|u_m\|_{L_\infty((0,T);H^2)}^2 \leq C \left(\|f\|_{L_\infty((0,T);H)}^2 + \|\ddot{u}_m\|_{L_\infty((0,T);H)}^2 \right).$$

From this and (4.9) we conclude

$$\begin{aligned} & \|\ddot{u}_m\|_{L_\infty((0,T);H)}^2 + \|\dot{u}_m\|_{L_\infty((0,T);V)}^2 + \|u_m\|_{L_\infty((0,T);H^2)}^2 \\ & \leq C\{\|u^0\|_{H^2}^2 + \|v^0\|_V^2 + \|f\|_{L_\infty((0,T);H)}^2 + \|\underline{f}\|_{L_1((0,T);H)}^2\}, \end{aligned}$$

that using $\|f\|_{L_\infty((0,T);H)} \leq C\|f\|_{W_1^1((0,T);H)}$, by Sobolev inequality, we have

$$\begin{aligned} & \|\ddot{u}_m\|_{L_\infty((0,T);H)}^2 + \|\dot{u}_m\|_{L_\infty((0,T);V)}^2 + \|u_m\|_{L_\infty((0,T);H^2)}^2 \\ & \leq C\{\|u^0\|_{H^2}^2 + \|v^0\|_V^2 + \|f\|_{W_1^1((0,T);H)}^2\} \\ & \leq C\{\|u^0\|_{H^2}^2 + \|v^0\|_V^2 + \|f\|_{H^1((0,T);H)}^2\}. \end{aligned}$$

Finally from (4.7), similar to the proof of Theorem 1, we obtain

$$\|\ddot{u}_m\|_{L_2((0,T);V^*)}^2 \leq C\{\|u^0\|_{H^2}^2 + \|v^0\|_V^2 + \|f\|_{H^1((0,T);H)}^2\}.$$

The last two estimates then, in the limit, imply (4.5) and the desired estimate (4.6). The proof is now complete. \square

Remark 4. If we continue differentiating (4.7) in time to investigate more regularity, we obtain

$$\begin{aligned} & \rho(\ddot{\underline{u}}_m(t), \varphi_k) + a(\dot{\underline{u}}_m(t), \varphi_k) - \sum_{i=1}^2 \int_0^t \beta_i(t-s) a_i(\dot{\underline{u}}_m(s), \varphi_k) ds \\ & = (\ddot{f}(t), \varphi_k) + \sum_{i=1}^2 \dot{\beta}_i(t) a_i(u_m(0), \varphi_k) \\ & + \sum_{i=1}^2 \beta_i(t) a_i(\underline{u}_m(0), \varphi_k), \quad k = 1, \dots, m, t \in (0, T), \end{aligned}$$

Further, from $\dot{\beta}_i(t)a_i(u_m(0), \varphi_k)$, $i = 1, 2$, we get $\dot{\beta}_i(t)a_i(u_m(0), \ddot{u}_m(t))$, but the $\dot{\beta}_i$ are not integrable. Besides, after integration in time, we can not use partial integration to transfer one time derivative from $\dot{\beta}$ to $\ddot{u}_m(t)$, since β is singular at $t = 0$. This means that we can not get more regularity with weakly singular kernels β_i . This also indicates that with smoother kernel we can get higher regularity in case of homogeneous Dirichlet boundary condition under the appropriate assumption on the data, that is, more regularity and compatibility conditions.

Remark 5. For the simplified problem (2.11), the inequality (4.11) is

$$\begin{aligned} \|Au_m(t)\|^2 &\leq \frac{2}{\epsilon} \left(\|f(t)\|^2 + \rho^2 \|\ddot{u}_m(t)\|^2 \right) + \epsilon \|Au_m(t)\|^2 \\ &\quad + \left(\int_0^t \beta(s) ds \right) \max_{0 \leq s \leq t} \|Au_m(s)\|^2. \end{aligned}$$

Hence, the assumption (4.4) can be ignored, since $\int_0^t \beta(s) ds < \gamma < 1$. That is, Theorem 3 applies also to the simplified problem (2.11), ignoring the assumption (4.4).

Remark 6. We recall the definition of the operators A, A_1 and A_2 from (2.5), and the fact that they are self-adjoint, positive definite linear operators. If the solution u is regular such that its second order partial derivatives are comutative, then the operator $A^{1/2}$ is comutative with the operators $A_1^{1/2}, A_2^{1/2}$. Here, the operator A^l ($l \in \mathbb{R}$) is defined by, see e.g., [19]

$$A^l v = \sum_{k=1}^{\infty} \lambda_k^l(v, \varphi_k) \varphi_k,$$

where $\{(\lambda_k, \varphi_k)\}_{k=1}^{\infty}$ are the eigenpairs of the operator A , and in a similar way A_1^l and A_2^l are defined. In this case the assumption (4.4) is replaced by

$$\int_0^t \max_{i=1,2} \beta_i(s) ds < 1,$$

since in (4.10) we have

$$\begin{aligned} \sum_{i=1}^2 \int_0^t \beta_i(t-s) a_i(u_m(s), Au_m(t)) ds &\leq \int_0^t \max_{i=1,2} \beta_i(t-s) a(u_m(s), Au_m(t)) ds \\ &\leq \left(\int_0^t \max_{i=1,2} \beta_i(t-s) ds \right) \max_{0 \leq s \leq t} \|Au_m(s)\|^2, \end{aligned}$$

where we used the fact that

$$\begin{aligned} a_i(v, Av) &= (A^{1/2} A_i^{1/2} A_i^{1/2} v, A^{1/2} v) \\ &= (A_i^{1/2} A^{1/2} A_i^{1/2} v, A^{1/2} v) = a_i(A^{1/2} v, A^{1/2} v) \geq 0. \end{aligned}$$

In the next theorem we prove regularity of any order of the solution of models with smooth kernels.

Theorem 4. *We assume that $\Gamma_N = \emptyset$, and condition (4.4) holds. Assume ($r = 0, 1, \dots$)*

$$u^0 \in H^{r+1}, \quad v^0 \in H^r, \quad \frac{d^k f}{dt^k} \in L_2((0, T); H^{r-k}) \quad (k = 0, \dots, r), \quad \beta_i \in W_1^{r-2}(0, T),$$

and the r^{th} -order compatibility conditions

$$\begin{aligned} u_0^0 &:= u^0 \in V, \quad u_1^0 := v^0, \\ u_2^0 &:= \frac{1}{\rho}(f(0) - Au^0) \in V, \quad \text{if } r = 2 \\ u_r^0 &:= \frac{1}{\rho} \left(\frac{d^{r-2}}{dt^{r-2}} f(0) - Au_{r-2}^0 + \sum_{j=0}^{r-3} \sum_{i=1}^2 \frac{d^j}{dt^j} \beta_i(0) A_i u_{r-3-j}^0 \right) \in V, \quad \text{if } r \geq 3. \end{aligned}$$

Then for the unique solution u of (2.8)-(2.10) we have

$$\frac{d^k}{dt^k} u \in L_\infty((0, T); H^{r+1-k}) \quad (k = 0, \dots, r+1),$$

and we have the estimate

$$\sum_{k=0}^{r+1} \left\| \frac{d^k u}{dt^k} \right\|_{L_\infty((0, T); H^{r+1-k})} \leq C \left(\sum_{k=0}^r \left\| \frac{d^k f}{dt^k} \right\|_{L_2((0, T); H^{r-k})} + \|u^0\|_{H^{r+1}} + \|v^0\|_{H^r} \right).$$

We note that, with $\beta_i \in W_1^{r-2}(0, T)$ we have $\beta_i \in C^{r-3}[0, T]$ by Sobolev inequality. Therefore u_r^0 in the compatibility conditions is well-defined.

Proof. The proof is by an induction, the case $r = 0$ following from Corollary 1 above.

Assume next the theorem valid for some $m \in \mathbb{N}$, and suppose

$$u^0 \in H^{r+2}, \quad v^0 \in H^{r+1}, \quad \frac{d^k f}{dt^k} \in L_2((0, T); H^{r+1-k}) \quad (k = 0, \dots, r+1).$$

Suppose also the $(r+1)^{\text{th}}$ -order compatibility conditions hold. Differentiating the model problem (2.6) with respect to t , we check that $\underline{u} := \dot{u}$ is the unique weak solution of

$$\begin{aligned} \rho \ddot{\underline{u}}(t) + A \underline{u}(t) \\ - \sum_{i=1}^2 \int_0^t \beta_i(t-s) A_i \underline{u}(s) ds &= \underline{f}(t) + \sum_{i=1}^2 \beta_i(t) A_i u^0 \quad \text{in } \Omega \times (0, T), \\ \underline{u} &= 0 \quad \text{on } \Gamma_D \times (0, T), \\ \underline{u}(\cdot, 0) = \underline{u}^0, \quad \dot{\underline{u}}(\cdot, 0) &= \underline{v}^0 \quad \text{in } \Omega, \end{aligned}$$

for

$$\underline{u}^0 = v^0, \quad \underline{v}^0 = \ddot{u}(0) = \frac{1}{\rho}(f(\cdot, 0) - Au^0), \quad \underline{f} = \dot{f}.$$

In particular, for $r = 0$ we rely upon Theorem 3 to be sure that $\underline{u} \in L_2((0, T); V)$, $\underline{u}' \in L_2((0, T); H)$, $\underline{u}'' \in L_2((0, T); V^*)$.

Since f, u^0, v^0 satisfy the $(r+1)^{\text{th}}$ -order compatibility conditions, it is easy to see that $\underline{f}, \underline{u}^0, \underline{v}^0$ satisfy the r^{th} -order compatibility conditions. Thus applying the induction assumption for \underline{u} we have

$$\frac{d^k \underline{u}}{dt^k} \in L_\infty((0, T); H^{r+1-k}) \quad (k = 0, \dots, r+1),$$

with the estimate

$$\sum_{k=0}^{r+1} \left\| \frac{d^k \underline{u}}{dt^k} \right\|_{L_\infty((0, T); H^{r+1-k})} \leq C \left(\sum_{k=0}^r \left\| \frac{d^k \underline{f}}{dt^k} \right\|_{L_2((0, T); H^{r-k})} + \|\underline{u}^0\|_{H^{r+1}} + \|\underline{v}^0\|_{H^r} \right).$$

Since $\underline{u} = \dot{u}$ we can write

$$\begin{aligned}
 & \sum_{k=1}^{r+2} \left\| \frac{d^k u}{dt^k} \right\|_{L_\infty((0,T);H^{r+2-k})} \\
 (4.12) \quad & \leq C \left(\sum_{k=1}^{r+1} \left\| \frac{d^k f}{dt^k} \right\|_{L_2((0,T);H^{r+1-k})} + \|v^0\|_{H^{r+1}} + \|Au^0\|_{H^r} + \|f(0)\|_{H^r} \right) \\
 & \leq C \left(\sum_{k=0}^{r+1} \left\| \frac{d^k f}{dt^k} \right\|_{L_2((0,T);H^{r+1-k})} + \|u^0\|_{H^{r+2}} + \|v^0\|_{H^{r+1}} \right),
 \end{aligned}$$

where, for the last inequality, we used the Sobolev inequality

$$\|f\|_{C([0,T];H^r)} \leq C(\|f\|_{L_2((0,T);H^r)} + \|\dot{f}\|_{L_2((0,T);H^r)}).$$

We now find a bound for $\|u\|_{L_\infty((0,T);H^{r+2})}$. We set $v = A^{r+1}u$ in (2.9) to obtain

$$a(u, A^{r+1}u) = (f - \rho \ddot{u}, A^{r+1}u) + \sum_{i=1}^2 \int_0^t \beta_i(t-s) a_i(u(s), A^{r+1}u(t)) ds.$$

This, using the Cauchy-Schwarz inequality and (2.7), implies

$$\begin{aligned}
 \|A^{\frac{r+2}{2}} u(t)\|^2 & \leq \frac{C}{\epsilon} \left(\|A^{r/2} f(t)\|^2 + \rho \|A^{r/2} \ddot{u}(t)\|^2 \right) + \epsilon \|A^{\frac{r+2}{2}} u(t)\|^2 \\
 & + \left(\sum_{i=1}^2 \int_0^t \beta_i(s) ds \right) \max_{0 \leq s \leq t} \|A^{\frac{r+2}{2}} u(s)\|^2,
 \end{aligned}$$

that gives us, by elliptic regularity (4.3),

$$\|u\|_{L_\infty((0,T);H^{r+2})}^2 \leq C \left(\|f\|_{L_\infty((0,T);H^r)}^2 + \|\ddot{u}\|_{L_\infty((0,T);H^r)}^2 \right).$$

Adding this inequality to (4.12) we deduce

$$\begin{aligned}
 & \sum_{k=0}^{r+2} \left\| \frac{d^k u}{dt^k} \right\|_{L_\infty((0,T);H^{r+2-k})} \\
 & \leq C \left(\sum_{k=0}^{r+1} \left\| \frac{d^k f}{dt^k} \right\|_{L_2((0,T);H^{r+1-k})} + \|u^0\|_{H^{r+2}} + \|v^0\|_{H^{r+1}} \right),
 \end{aligned}$$

that is the assertion of the theorem for $r+1$. Now the proof is complete. \square

We note that Remark 5 also holds for Theorem 4. Remark 6 can be applied to Theorem 4, provided the solution u is smooth enough such that the operator $A^{\frac{r+1}{2}}$ is comutative with the operators $A_i^{1/2}$, $i = 1, 2$, that is, when $(r+2)$ -th order partial derivatives of the solution u are comutative.

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